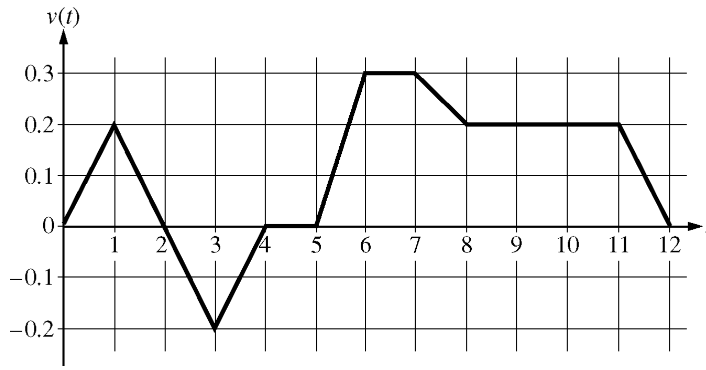
AP® CALCULUS BC
2009 SCORING GUIDELINES

Question 1



Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

(a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|\, dt$.

(c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

(d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function w given by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

(a) $a(7.5) = v'(7.5) = \dfrac{v(8) - v(7)}{8 - 7} = -0.1$ miles/minute^2	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{units} \end{cases}$
(b) $\int_0^{12} \lvert v(t)\rvert\, dt$ is the total distance, in miles, that Caren rode during the 12 minutes from $t = 0$ to $t = 12$. $\int_0^{12} \lvert v(t)\rvert\, dt = \int_0^{2} v(t)\, dt - \int_2^{4} v(t)\, dt + \int_4^{12} v(t)\, dt$ $= 0.2 + 0.2 + 1.4 = 1.8$ miles	$2 : \begin{cases} 1 : \text{meaning of integral} \\ 1 : \text{value of integral} \end{cases}$
(c) Caren turns around to go back home at time $t = 2$ minutes. This is the time at which her velocity changes from positive to negative.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$
(d) $\int_0^{12} w(t)\, dt = 1.6$; Larry lives 1.6 miles from school. $\int_0^{12} v(t)\, dt = 1.4$; Caren lives 1.4 miles from school. Therefore, Caren lives closer to school.	$3 : \begin{cases} 2 : \text{Larry's distance from school} \\ \quad 1 : \text{integral} \\ \quad 1 : \text{value} \\ 1 : \text{Caren's distance from school} \\ \quad \text{and conclusion} \end{cases}$

1A1

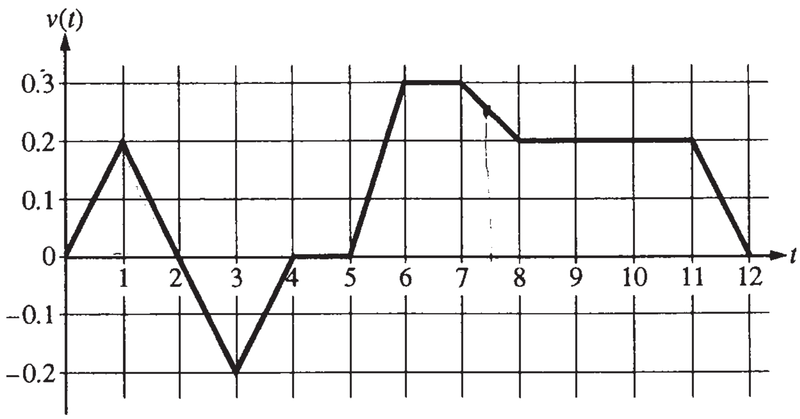
CALCULUS BC

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

$$a(7.5) = v'(7.5) = \frac{v(8) - v(7)}{8-7} = 0.2 - 0.3 = -0.1$$

$\therefore\ a(7.5) = -0.1\ \text{mi}/\text{min}^2$

Work for problem 1(b)

$$\int_0^{12} |v(t)|\,dt = 0.2 + 0.2 + 0.15 + 0.3 + 0.2 + 0.05 + 6(0.1) + 0.1$$
$$= 1.8 \text{ miles}$$

$\therefore\ \int_0^{12} |v(t)|\,dt$ is the total distance that Caron traveled from time $t = 0$ min to $t = 12$ min to arrive to the school, which is 1.8 mi.

Do not write beyond this border.

Continue problem 1 on page 5

1A₂:

Do not write beyond this border.

Work for problem 1(c)

She turns around at $t=2$ minutes because that is when her velocity changes from positive to negative.

Work for problem 1(d)

$\int_0^{12} w(t)\,dt = \int_0^{12} \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)dt = 1.6$ mi

The distance from Larry's house to school : 1.6mi

$\int_0^{12} v(t)\,dt = 0.15+0.3+0.2+0.05+0.6+0.1 = 1.4$ mi

The distance from Caren's house to school : 1.4mi

∴ Caren lives closer to school because the distance from school to her house is smaller than that to Larry's house.

1B1

1 1 1 1 1 1 1 1 1 1

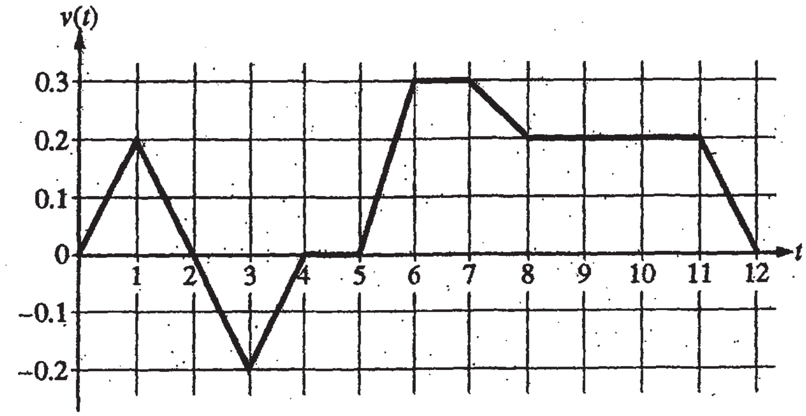
CALCULUS AB

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Do not write beyond this border.

Work for problem 1(a)

$a(t) = \frac{dv}{dt}$ $\frac{v_f - v_i}{t_f - t_i}$ $\frac{.2 - .3}{8-7}$

$\frac{-.1}{1}$ $-.1$

$-.1$ miles per minute

Work for problem 1(b)

$\int_0^{12} |v(t)|\,dt$ would show the total distance in miles that Caren traveled in 12 minutes

$\int_0^2 v(t)dt - \int_2^4 v(t)dt \int_4^{12} v(t)dt$ $-.3/2 + .11 + -.1 + .05$

$\cancel{.2}$ $-.2$ $+.41$ $.81$

Continue problem 1 on pag

1Bz

Work for problem 1(c)

$t=2$ because her velocity changes from $+$ to $-$ and the $\int_0^2 |v(t)|\,dt = \int_2^4 |v(t)|\,dt$

Do not write beyond this border.

Work for problem 1(d)

$w(t) = \frac{\pi}{15} \sin\left(\frac{\pi}{12} t\right)$

Caren $\int_0^{12} |v(t)|\,dt = .81$ miles

Larry $\int_0^{12} w(t)\,dt = 1.6$ miles

Caren lives closer to school because she traveled less distance to get there

GO ON TO THE NEXT PA

1C1

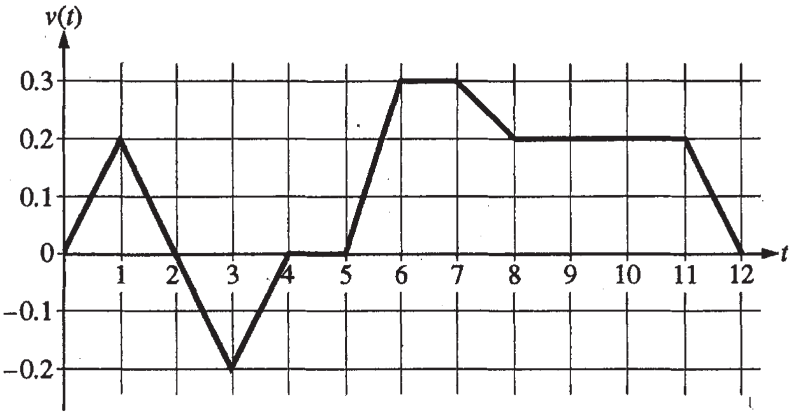
CALCULUS BC

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Do not write beyond this border.

Work for problem 1(a)

acceleration $= v'(t)$.

$v'(7.5)$

Work for problem 1(b)

$\int$ of velocity is the position.

so, $\int_0^{12}|v(t)|dt$ means ~~the~~ how much or how far of distance Caren rode ~~by~~ bicycle in 12 minutes.

$\int_0^{12}|v(t)|dt = \cancel{\frac{(1\times 0.2)}{2}+\frac{(1\times 0.2)}{2}+}$

↑ signed area. $\left[\left[\frac{(1\times 0.2)}{2}\times 4\right]+\left(\frac{1\times 0.3}{2}\right)+(1\times 0.3)+(1\times 0.2)+\left(\frac{1\times 0.1}{2}\right)+\left((1\times 0.2)\times 3\right)+\left(\frac{1\times 0.2}{2}\right)\right] = 1.8$

Continue problem 1 on page 5

Do not write beyond this border.

Work for problem 1(c)

~~She~~ She turns around to go back home at $t = 2$. the graph in the interval $1 \le x \le 3$, changes the direction.

~~She was~~

Work for problem 1(d)

~~At $t=6$, $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{15} = 0.20930$~~
~~Larry's position from home~~

Larry's ~~po~~ distances to school : $\int_0^{12} \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)dt = 1.6$.

Caren's " $= \int_0^{12} f(t)\,dt = 1.8$ (from answer (b))

so. Larry lives closer to school.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

AP® CALCULUS BC
2009 SCORING COMMENTARY

Question 1

Overview

This problem opened with a piecewise-linear graph. The graph models the velocity function $v(t)$ for bicycle rider Caren during a 12-minute period in which she travels along a straight road, starting at home at time $t = 0$ and arriving at school at time $t = 12$. Part (a) asked for Caren's acceleration at a particular time during her trip, which required students to recognize that acceleration is the derivative of velocity and to acquire the value of this derivative from the slope of the appropriate line segment on the given velocity graph. Part (b) asked for an interpretation of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip, as well as for the value of this integral. Part (c) provided the additional information that Caren needed to return home to retrieve her homework shortly after starting her journey. Students needed to associate Caren's direction of motion with the sign of her velocity to determine at what time she turned around. (Students were not required to observe that the distances traveled in each direction match.) In part (d) the velocity function for another bicycle rider, Larry, was modeled by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$ for the same 12-minute period, $0 \le t \le 12$. This part asked who lives closer to school, Caren or Larry. To respond, students needed to compute the two home-to-school distances, $\int_0^{12} v(t)\, dt$ (which equals $\int_5^{12} v(t)\, dt$) and $\int_0^{12} w(t)\, dt$.

Sample: 1A
Score: 9

The student earned all 9 points.

Sample: 1B
Score: 6

The student earned 6 points: 1 point in part (a), 1 point in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the student earned the first point for evaluating a correct difference quotient. The student's units of miles per minute are incorrect. In part (b) the student earned the first point for a correct interpretation of the meaning of the integral using correct units. The student's evaluation of the integral is incorrect. In part (c) the student's work is correct. The statement regarding the integrals of $|v(t)|$ on the two different intervals is correct but was not required to earn the point. In part (d) the student earned 2 points for Larry's distance from school by stating and evaluating the correct definite integral. The student's value for Caren's distance from school is incorrect, so the last point was not earned.

Sample: 1C
Score: 4

The student earned 4 points: no points in part (a), 1 point in part (b), 1 point in part (c), and 2 points in part (d). In part (b) the student does not indicate the correct units of miles, so the first point was not earned. The student earned the second point for a correct evaluation of the integral. In part (c) the student earned the first point for a correct answer. The student's reason is not valid. In part (d) the student earned 2 points for Larry's distance from school by stating and evaluating the correct definite integral. The student's value for Caren's distance from school is incorrect, so the last point was not earned.

AP® CALCULUS BC
2009 SCORING GUIDELINES

Question 2

The rate at which people enter an auditorium for a rock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; $R(t)$ is measured in people per hour. No one is in the auditorium at time $t = 0$, when the doors open. The doors close and the concert begins at time $t = 2$.

(a) How many people are in the auditorium when the concert begins?

(b) Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.

(c) The total wait time for all the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The function w models the total wait time for all the people who enter the auditorium before time t. The derivative of w is given by $w'(t) = (2 - t)R(t)$. Find $w(2) - w(1)$, the total wait time for those who enter the auditorium after time $t = 1$.

(d) On average, how long does a person wait in the auditorium for the concert to begin? Consider all people who enter the auditorium after the doors open, and use the model for total wait time from part (c).

(a) $\int_0^2 R(t)\,dt = 980$ people	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(b) $R'(t) = 0$ when $t = 0$ and $t = 1.36296$ The maximum rate may occur at 0, $a = 1.36296$, or 2. $R(0) = 0$ $R(a) = 854.527$ $R(2) = 120$ The maximum rate occurs when $t = 1.362$ or 1.363.	$3 : \begin{cases} 1 : \text{considers } R'(t) = 0 \\ 1 : \text{interior critical point} \\ 1 : \text{answer and justification} \end{cases}$
(c) $w(2) - w(1) = \int_1^2 w'(t)\,dt = \int_1^2 (2 - t)R(t)\,dt = 387.5$ The total wait time for those who enter the auditorium after time $t = 1$ is 387.5 hours.	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(d) $\frac{1}{980}w(2) = \frac{1}{980}\int_0^2 (2 - t)R(t)\,dt = 0.77551$ On average, a person waits 0.775 or 0.776 hour.	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

Do not write beyond this border.

Work for problem 2(a)

$\int_0^2 R(t)\,dt = \boxed{980 \text{ people}}$

Work for problem 2(b)

$R'(t)=0$
$t=1.3629=A$

Endpoints
$t=0$
$t=2$

R has an absolute maximum at $t=1.3629$ hours on $t\in[0,2]$ guaranteed by the EVT.

t	$R(t)$
0	0
A	854.5273
2	120

Continue problem 2 on page 7.

Do not write beyond this border.

Work for problem 2(c)

$\int_1^2 w'(t)\,dt = \boxed{387.5 \text{ hours}}$

Work for problem 2(d)

$\int_0^2 w'(t)\,dt \div \int_0^2 R(t)\,dt = \frac{760}{980} = \boxed{0.7755 \frac{\text{hours}}{\text{person}}}$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Do not write beyond this border.

Work for problem 2(a)

People $= \int_0^2 R(t)\,dt = 980$ people

There are approximately 980 people in the auditorium when the concert begins.

Work for problem 2(b)

$R(t) = 1380t^2 - 675t^3$

$R'(t) = 2760t - 2025t^2$

$0 = 2760t - 2025t^2$

$t = 0, 1.363$

$R'(t)$ 0 0

-1 0 1 1.363 2

$R'(-1) = -4785$ $R'(2) = -2580$

$R'(1) = 735$

The rate at which people enter the auditorium is at a max at $t = 1.363$ hour b/c $R'(t) = 0$ at $t = 1.363$ hours & changes from + to -

Continue problem 2 on page 7.

Do not write beyond this border.

Work for problem 2(c)

$\int_a^b w'(t) = w(b) - w(a)$

$w'(t) = (2-t)R(t)$

$w(2) - w(1) = \int_1^2 (w'(t))dt$

$= \int_1^2 (2-t)(1380t^2 - 675t^3)dt$

$= 387.5$ hours

Total wait time is 387.5 hours according to the fundamental theorem of Calculus

Work for problem 2(d)

Avg. wait time $= \frac{1}{b-a}\int_a^b w'(t)dt$

$= \frac{1}{2-1}\int_1^2 w'(t)dt$

$= \frac{1}{1}\int_1^2 w'(t)dt$

$= 387.5$

On average a person waits 387.5 hours in the auditorium for conart to begin

GO ON TO THE NEXT PAGE.

Do not write beyond this border.

Work for problem 2(a)

$$\int_0^2 1380t^2 - 675t^3\,dt$$

$$1380 \cdot \frac{1}{3} t^3 - 675 \cdot \frac{1}{4} t^4$$

$$460t^3 - 168.75t^4$$

$$\left(460(2)^3 - 168.75(2)^4\right) - \left(460(0)^3 - 168.75(0)^4\right)$$

$980 - 0 = 980$ people are in the auditorium when the concert begins

Work for problem 2(b)

$$R(t) = 1380t^2 - 675t^3$$

$$R'(t) = 2760t - 2025t^2$$

$$2760t - 2025t^2 = 0$$

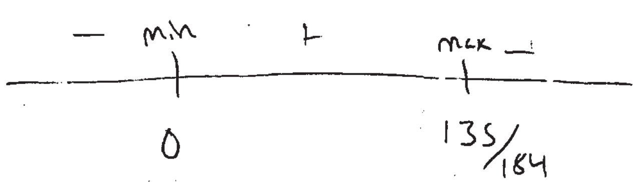
$$t(2760 - 2025t) = 0$$

$$t = 0, \tfrac{135}{184}$$

$$2760 - 2025t = 0$$

$$-2025t = -2760$$

$$t = \tfrac{135}{184}$$



The rate at which people enter the auditorium is at its maximum at time $\frac{135}{184}$.

Continue problem 2 on page 7.

Work for problem 2(c)

$w'(t) = (2-t)\, R(t)$

$w'(t) = (2-t)(1380t^2 - 675t^3)$

$\int_1^2 (2-t)(1380t^2 - 675t^3)\, dt =$ ______ hours

Do not write beyond this border.

Work for problem 2(d)

$$\frac{\text{\_\_\_\_\_\_ hours = the total wait time}}{\text{980 people who are in the auditorium when the concert begins}}$$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

AP® CALCULUS BC
2009 SCORING COMMENTARY

Question 2

Overview

This problem presented students with a polynomial $R(t) = 1380t^2 - 675t^3$ that modeled the rate, in people per hour, at which people enter an auditorium during the two hours ($0 \le t \le 2$) prior to the start of a rock concert. It was stated that the auditorium was empty at time $t = 0,$ and part (a) asked for the number of people in the auditorium at time $t = 2,$ which required computation of the definite integral $\int_0^2 R(t)\, dt.$ In part (b) students needed to find the time t that maximizes $R(t).$ Part (c) defined the total wait time for all the people in the auditorium and stated that a function w that models the total wait time for all the people who entered the auditorium by time t has derivative $w'(t) = (2 - t)R(t)$. Students were asked to evaluate $w(2) - w(1)$ and should have recognized that this is computed by $\int_1^2 w'(t)\, dt.$ Part (d) asked for the average amount of time that a concertgoer spent waiting for the concert to begin after entering the auditorium. Students needed to compute the total wait time, $\int_0^2 w'(t)\, dt,$ for all people attending the concert and divide this by the number of people in the auditorium at the start of the concert as found in part (a).

Sample: 2A
Score: 9

The student earned all 9 points.

Sample: 2B
Score: 6

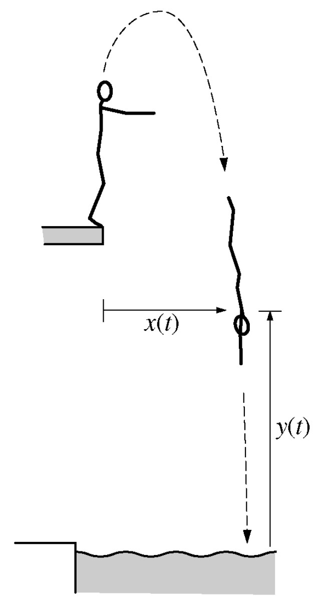
The student earned 6 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student earned the first 2 points by correctly computing $R'(t)$ and determining the correct interior critical point. The student considers the sign change of R' at $t = 1.363,$ providing an argument for a local maximum instead of a global maximum, and did not earn the third point. In part (c) the student's work is correct. In part (d) the student computes the average value of $w'(t)$ over the interval from 1 to 2 instead of the total wait time $w(2)$ divided by the total number of people.

Sample: 2C
Score: 4

The student earned 4 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student earned the first point for correctly computing $R'(t)$. The student's value for the critical point is incorrect, so the response was not eligible for the third point. In this case, no justification for a global maximum is given. In part (c) the student earned the first point for providing the correct definite integral for $w(2) - w(1).$ The student does not compute the value of the integral. In part (d) the student does not provide a definite integral for the numerator.

AP® CALCULUS BC
2009 SCORING GUIDELINES

Question 3



Note: Figure not drawn to scale.

A diver leaps from the edge of a diving platform into a pool below. The figure above shows the initial position of the diver and her position at a later time. At time t seconds after she leaps, the horizontal distance from the front edge of the platform to the diver's shoulders is given by $x(t)$, and the vertical distance from the water surface to her shoulders is given by $y(t)$, where $x(t)$ and $y(t)$ are measured in meters. Suppose that the diver's shoulders are 11.4 meters above the water when she makes her leap and that

$$\frac{dx}{dt} = 0.8 \text{ and } \frac{dy}{dt} = 3.6 - 9.8t,$$

for $0 \le t \le A$, where A is the time that the diver's shoulders enter the water.

(a) Find the maximum vertical distance from the water surface to the diver's shoulders.

(b) Find A, the time that the diver's shoulders enter the water.

(c) Find the total distance traveled by the diver's shoulders from the time she leaps from the platform until the time her shoulders enter the water.

(d) Find the angle θ, $0 < \theta < \frac{\pi}{2}$, between the path of the diver and the water at the instant the diver's shoulders enter the water.

(a) $\frac{dy}{dt} = 0$ only when $t = 0.36735$. Let $b = 0.36735$.

The maximum vertical distance from the water surface to the diver's shoulders is

$$y(b) = 11.4 + \int_0^b \frac{dy}{dt}\, dt = 12.061 \text{ meters.}$$

Alternatively, $y(t) = 11.4 + 3.6t - 4.9t^2$, so $y(b) = 12.061$ meters.

$3 : \begin{cases} 1 : \text{considers } \frac{dy}{dt} = 0 \\ 1 : \text{integral or } y(t) \\ 1 : \text{answer} \end{cases}$

(b) $y(A) = 11.4 + \int_0^A \frac{dy}{dt}\, dt = 11.4 + 3.6A - 4.9A^2 = 0$ when

$A = 1.936$ seconds.

$2 : \begin{cases} 1 : \text{equation} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^A \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 12.946$ meters

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) At time A, $\frac{dy}{dx} = \left.\frac{dy/dt}{dx/dt}\right|_{t=A} = -19.21913.$

The angle between the path of the diver and the water is

$\tan^{-1}(19.21913) = 1.518$ or 1.519.

$2 : \begin{cases} 1 : \frac{dy}{dx} \text{ at time } A \\ 1 : \text{answer} \end{cases}$

Do not write beyond this border.

Work for problem 3(a)

$3.6 - 9.8t = 0$

$t = 0.367$

$11.4 + \int_0^{0.367} 3.6 - 9.8t \, dt \approx \boxed{12.061 \text{ meters}}$

Work for problem 3(b)

$11.4 + \int_0^{A} 3.6 - 9.8t \, dt = 0$

$\int_0^{A} 3.6 - 9.8t \, dt = -11.4$

$A \approx \boxed{1.936 \text{ sec}}$

Do not write beyond this border.

Continue problem 3 on page 9.

3A2

Work for problem 3(c)

$$\int_0^{1.936} \sqrt{(0.8)^2 + (3.6 - 9.8t)^2}\, dt \approx \boxed{12.946 \text{ meters}}$$

Work for problem 3(d)

$$\frac{dy}{dx} = \frac{3.6 - 9.8t}{0.8}$$

$$\left.\frac{dy}{dx}\right|_{t=1.936} \approx -19.219$$

$$\theta = |\arctan(-19.219)| \approx \boxed{1.519}$$

END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Do not write beyond this border.

3B₁

Work for problem 3(a)

$$\frac{dy}{dt} = 0 = 3.6 - 9.8t$$

$$t = .3673469388$$

$$11.4 + \int_0^{.3673469388} (3.6 - 9.8t)\,dt = 12.06122449 \text{ meters}$$

Do not write beyond this border.

Work for problem 3(b)

$$11.4 + \int_0^A (3.6 - 9.8t)\,dt = 0$$

$$11.4 + \Big|_0^A (3.6t - 4.9t^2) = 0$$

$$11.4 + 3.6A - 4.9A^2 = 0$$

$$A = 1.93626 \text{ seconds}$$

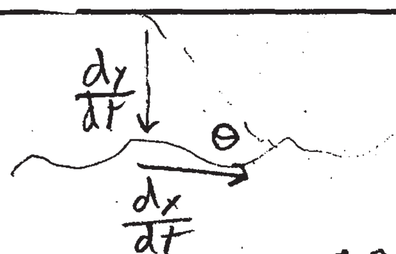
Continue problem 3 on page 9.

3B2

Work for problem 3(c)

$$12.06122449 + \int_0^{.3673469388} (3.6 - 9.8t)\,dt = 12.72245 \text{ meters}$$

Work for problem 3(d)



$$\tan\theta = \frac{\frac{dy}{dt}(A)}{\frac{dx}{dt}(A)} = \frac{3.6 - 9.8A}{0.8} = -19.219185$$

$$\theta = \arctan -19.219185 = -1.5708$$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Do not write beyond this border.

Do not write beyond this border.

3C1

Work for problem 3(a)

$\frac{dy}{dt} / \frac{dx}{dt} = \frac{dy}{dx} = \frac{3.6 - 9.8t}{.8}$

max at $\frac{dy}{dt} = 3.6 - 9.8t = 0$

$\boxed{t = .367 \text{sec}}$

Work for problem 3(b)

enters water at $y=0$.

$\int \frac{dy}{dt} = 3.6t - 4.9t^2 + 11.4 = 0$

$\boxed{t = 1.93626}$ sec

Do not write beyond this border.

Do not write beyond this border.

Continue problem 3 on page 9.

 $3C_2$

Work for problem 3(c)

arc length $= \int_a^b \sqrt{(x'(t))^2 + y'(t)^2}\, dt$

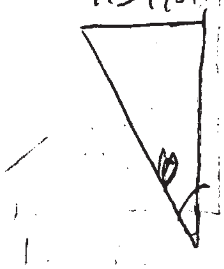
$$\int_0^{1.93626} \sqrt{(.8)^2 + (3.6-9.8t)^2}\, dt = \boxed{12.9463 \text{ meters}} + 11.4 \text{ meters}$$

$$= \boxed{24.346 \text{ meters}}$$

Work for problem 3(d)

at $t = 1.93626$ – when the diver enters the water – $\int \frac{dx}{dt}dt = .8t$ & $\int \frac{dy}{dt} = 3.6t - 4.9t^2 + 11.4$

at $t = 1.93626$, $x(t) = 1.54901$, $y(t) = 16.5333$



1.54901

16.5333

$\tan\theta = \frac{1.54901}{16.5333} = .093689$

$\tan^{-1}(.093689) =$

$\boxed{.0934 \text{ radians}}$

END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Do not write beyond this border.

Do not write beyond this border.

Question 3

Overview

This problem described the path of a diver's shoulders during a dive from a platform into a pool, using parametric functions $x(t)$ for the horizontal distance from the edge of the platform and $y(t)$ for the vertical distance from the water. It was stated that the diver's shoulders were 11.4 meters above the water at the start, and that $\frac{dx}{dt} = 0.8$ and $\frac{dy}{dt} = 3.6 - 9.8t$ for $0 \le t \le A$, where A is the time that the diver's shoulders enter the water. Part (a) asked for the maximum height of the diver's shoulders above the water. Students needed to use the given derivative of y to determine when the diver's shoulders were highest and then to combine the initial height of the shoulders above the water with the appropriate integral to determine the maximum height. Part (b) asked for the time A that the diver's shoulders enter the water. This involved integrating to find the height of the shoulders above the water at time A and then solving for when this height was zero. Part (c) asked for the total distance the diver's shoulders traveled during the dive, which required an arclength calculation for the curve described by $x(t)$ and $y(t)$ from time 0 to the time A found in part (b). Part (d) asked for the acute angle between the path of the diver and the water's surface at time A. Students needed to find the slope of the path, $\frac{dy}{dx}$, at time A and then solve for the angle by realizing that its tangent matches the absolute value of this slope.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: 3 points in part (a), 2 points in part (b), no points in part (c), and 1 point in part (d). In parts (a) and (b) the student's work is correct. In part (c) the student's definite integral does not represent a calculation of arclength. In part (d) the student correctly evaluates the ratio $\frac{dx/dt}{dy/dt}$ at the value of A and earned the first point. The student presents a value for θ that is not in the interval $0 < \theta < \frac{\pi}{2}$ and did not earn the second point.

Sample: 3C
Score: 4

The student earned 4 points: 1 point in part (a), 2 points in part (b), 1 point in part (c), and no points in part (d). In part (a) the student considers $\frac{dy}{dt} = 0$ and earned the first point. In part (b) the student's work is correct. In part (c) the student earned the first point for setting up the correct definite integral for the calculation of arclength. The integral is evaluated correctly, but the student incorrectly adds the initial distance of the diver at $t = 0$. The answer point was not earned. In part (d) the student incorrectly calculates the ratio of distances $\frac{y(t)}{x(t)}$ at A instead of $\frac{dy}{dx}$ at A and was not eligible for any points.

Question 4

Consider the differential equation $\frac{dy}{dx} = 6x^2 - x^2 y$. Let $y = f(x)$ be a particular solution to this differential equation with the initial condition $f(-1) = 2$.

(a) Use Euler's method with two steps of equal size, starting at $x = -1$, to approximate $f(0)$. Show the work that leads to your answer.

(b) At the point $(-1, 2)$, the value of $\frac{d^2y}{dx^2}$ is -12. Find the second-degree Taylor polynomial for f about $x = -1$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

(a)
$$f\left(-\frac{1}{2}\right) \approx f(-1) + \left(\left.\frac{dy}{dx}\right|_{(-1,\,2)}\right) \cdot \Delta x$$
$$= 2 + 4 \cdot \frac{1}{2} = 4$$

$$f(0) \approx f\left(-\frac{1}{2}\right) + \left(\left.\frac{dy}{dx}\right|_{\left(-\frac{1}{2},\,4\right)}\right) \cdot \Delta x$$
$$\approx 4 + \frac{1}{2} \cdot \frac{1}{2} = \frac{17}{4}$$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$

(b) $P_2(x) = 2 + 4(x+1) - 6(x+1)^2$

1 : answer

(c)
$$\frac{dy}{dx} = x^2(6 - y)$$
$$\int \frac{1}{6 - y}\, dy = \int x^2\, dx$$
$$-\ln|6 - y| = \frac{1}{3}x^3 + C$$
$$-\ln 4 = -\frac{1}{3} + C$$
$$C = \frac{1}{3} - \ln 4$$
$$\ln|6 - y| = -\frac{1}{3}x^3 - \left(\frac{1}{3} - \ln 4\right)$$
$$|6 - y| = 4e^{-\frac{1}{3}\left(x^3+1\right)}$$
$$y = 6 - 4e^{-\frac{1}{3}\left(x^3+1\right)}$$

$6 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Do not write beyond this border.

Work for problem 4(a)

Let Δx = step size = .5

$f(-.5) = f(-1) + (6(-1)^2 - (-1)^2(2)).5$

$= 2 + (6-2).5 = 2+2 = 4 = f(-.5)$

$f(0) \approx f(-.5) + (6(-.5)^2 - (-.5)^2(4)).5$

$= 4 + (6(\frac{1}{4}) - (\frac{1}{4})(4))(.5)$

$= 4 + (\frac{6}{4} - \frac{4}{4})(\frac{1}{2}) = 4 + (\frac{1}{2})(\frac{1}{2}) = \boxed{4 + \frac{1}{4}}$

Work for problem 4(b)

$\left.\frac{dy}{dx}\right]_{(-1,2)} = 6(1) - (1)(2) = 4$

$\left.\frac{d^2y}{dx^2}\right]_{(-1,2)} = -12$

$f(-1) = 2$

$\boxed{P(x) = 2 + 4(x+1) - \frac{12(x+1)^2}{2}} = 2 + 4(x+1) - 6(x+1)^2$

Continue problem 4 on page

NO CALCULATOR ALLOWED

Work for problem 4(c)

$$\frac{dy}{dx} = 6x^2 - x^2y$$

$$\frac{dy}{dx} = x^2(6-y)$$

$$\int \frac{dy}{(6-y)} = \int x^2\,dx$$

$$-\ln|6-y| = \frac{x^3}{3} + C$$

$$\ln|6-y| = \frac{-x^3}{3} + C$$

Using point $(-1,2)$

$$\ln(4) = \frac{1}{3} + C$$

$$C = \ln 4 - \frac{1}{3}$$

$$\ln|6-y| = \frac{-x^3}{3} + \ln 4 - \frac{1}{3}$$

$$6-y = e^{-x^3/3 + \ln 4 - \frac{1}{3}}$$

$$\boxed{y = -e^{-\frac{x^3}{3} + \ln 4 - \frac{1}{3}} + 6}$$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

4B.

4 4 4 4 4 4 4 4 4 4

NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Work for problem 4(a)

$\frac{dy}{dx} = 6x^2 - x^2 y$ $f(x) = y$ $f(-1) = 2$

$y_{new} = \left(\frac{dy}{dx}\right)(\text{step size}) + y_{old}$ step size $= .5$

$(6x^2 - x^2 y)(\cancel{-}.5) + y_{old}$

so $\boxed{\cancel{so}\ f(0) = 4.25}$

step 1 $6(-1)^2 - (-1)^2(2)(.5) + 2 = 4$

$(6-2)(.5)$

$2 + 2 = 4$

so at $x = \cancel{-}.5$ $y = 4$

step 2 $(6(-.5)^2 - (-.5)^2(4))(\cancel{+.5}) + \cancel{4} =$

$(6(.25) - (.25)(4))(\cancel{.5}) + 4$

$\left(\frac{6}{4} - \cancel{\frac{4}{4}}\right)(.5) + 4 =$

$\frac{1}{2}(.5) + 4 =$

$.25 + 4 = 4.25$

Work for problem 4(b)

$\frac{\cancel{dy}}{dx} = 6x^2 - x^2 y$

$x^2(6-y)$

$\frac{dy}{6-y} = x^2\,dx$

$-\int \frac{-dy}{-y+6} = \int x^2 dx$

$-\ln|y+6| = \frac{x^3}{3} + C$

$y+6 = e^{-\frac{x^3}{3}} + C$

$y = e^{-\frac{x^3}{3}} + C$

$f(-1) = 2$

$f'(-1) = 4$

$f''(-1) = -12$

$\frac{f^{(n)}(c)}{n!}(x-c)^{(n)}$

$6(-1)^2 - (-1)^2(2) = f'(-1)$

$6 - 2 = 4$

$P(x) = 4(x+1)^1 - 6(x+1)^2$

Do not write beyond this border.

4B2

4 4 4 4 4 4 4 4 4 4

NO CALCULATOR ALLOWED

Work for problem 4(c)

$y = f(x)$

$$\frac{dy}{dx} = 6x^2 - x^2y$$
$$= x^2(6-y)$$

$$\frac{dy}{6-y} = x^2 dx$$

$$-\int \frac{dy}{-y+6} = \int x^2 dx$$

$$-\ln|y+6| = \frac{x^3}{3} + C$$

$$y+6 = e^{-\frac{x^3}{3}} + C$$

$$y = e^{-\frac{x^3}{3}} + C$$

$$2 = e^{-\frac{(-1)^3}{3}} + C$$

$$2 = e^{\frac{1}{3}} + C$$

$$\ln 2 = \frac{1}{3} + C$$

$$\ln 2 - \frac{1}{3} = C$$

or

$$2 - e^{\frac{1}{3}} = C$$

$$y = e^{-\frac{x^3}{3}} + \left(\ln 2 - \frac{1}{3}\right)$$

or

$$y = e^{-\frac{x^3}{3}} + \left(2 - e^{\frac{1}{3}}\right)$$

Do not write beyond this border.

4C1

4 4 4 4 4 4 4 4 4 4

NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

Do not write beyond this border.

Work for problem 4(a)

$\frac{dy}{dx} = 6x^2 - x^2 y$

$f(-1) = 2$

$6(-1)^2 - (-1)^2(2)$

$(6) - (2) = 4$

$.25 \times 6 = 1.50$

$.5 \times .5 = .25$

$.25 \times 2 = .50$

$6(-0.5)^2 - (-0.5)^2(2)$

$(6)(0.25) - (0.25)(2)$

$(1.50) - (.50)$

$= 1$

$y - y_1 = m(x - x_1)$

$y - 2 = 4(-.5 + 1)$

$y - 2 = 2$

$y = 2$

$(-0.5, 2)$

$y - 2 = 1(0 + (-0.5))$

$y - 2 = 0.5$

$+2 \quad +2$

$y = 2.5$

$\boxed{f(0) = 2.5}$

Work for problem 4(b)

$(-1, 2)$ $\frac{d^2y}{dx^2} = -12$

Continue problem 4 on page 1

NO CALCULATOR ALLOWED

Work for problem 4(c)

x y
$f(-1)=2$

$$\frac{dy}{dx} = 6x^2 - x^2 y$$

$$dy = 6x^2 - x^2 y \, dx$$

$$\frac{dy}{y} = 6x^2 - x^2 dx$$

$$\int \frac{dy}{y} = \int 6x^2 - x^2 \, dx$$

$$\ln y = \frac{6x^3}{3} - \frac{x^3}{3} + C$$

$$e^{\ln y} = e^{2x^3 - \frac{x^3}{3}} + C$$

$$y = e^{2x^3 - \frac{x^3}{3}} + C$$

$$y = e^{2x^3 - \frac{x^3}{3}} \cdot e^C$$

$-1 \cdot -1 = 1 \cdot -1 = -1$

$$y = C \cdot e^{2x^3 - x^3/3}$$

$$2 = C \cdot e^{2(-1)^3 - (-1)^3/3}$$

$\frac{3(-2)}{3 \cdot 1} - \left(\frac{-1}{3}\right)$

$$2 = C \cdot e^{(-2) - (-1/3)}$$

$$2 = Ce^{-5/3}$$

$\frac{(-6) - (-1)}{3}$

$\frac{-6+1}{3} = \frac{-5}{3}$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 4

Overview

This problem presented the differential equation $\frac{dy}{dx} = 6x^2 - x^2 y$ and a particular solution $y = f(x)$ satisfying $f(-1) = 5$. Part (a) asked students to use Euler's method with two steps of equal size to approximate $f(0)$. In part (b) it was stated that $\left.\frac{d^2y}{dx^2}\right|_{(-1,\,2)} = -12$, and students were asked to provide the second-degree Taylor polynomial for f about $x = -1$. Part (c) asked for the particular solution $y = f(x)$.

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 2 points in part (a), no points in part (b), and 4 points in part (c). In part (a) the student's work is correct. In part (b) the student neglects to include the constant term in the polynomial. In part (c) the student earned the first point for separating the variables correctly. The student earned only 1 of the antiderivative points since the antidifferentiation of the expression that results in the logarithm is incorrect. The student earned the constant of integration and initial condition points. The student makes errors in attempting to find the specific constant of integration and was not eligible for the last point since the logarithm of a difference is not included (either $\ln|6 - y|$, $\ln(6 - y)$, or $\ln|y - 6|$.)

Sample: 4C
Score: 4

The student earned 4 points: 1 point in part (a), no points in part (b), and 3 points in part (c). In part (a) the student earned the first point for indicating two steps of Euler's method. The student makes an arithmetic error in calculating the y-coordinate in the first step and did not earn the answer point. In part (c) the student did not earn the separation of variables point. Because the separation results in the reciprocal of a linear function in y and a nontrivial function of x, the student was eligible for both antiderivative points. In this case, the student earned only 1 of the antiderivative points since the absolute value is required for all other arguments of the logarithmic function except $6 - y$. The student earned the constant of integration and initial condition points. The student was not eligible for the last point since the antiderivative in y does not result in $\ln(6 - y)$ or $\ln(y - 6)$.

Question 5

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

(a) Estimate $f'(4)$. Show the work that leads to your answer.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\, dx$. Show the work that leads to your answer.

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\, dx$. Show the work that leads to your answer.

(d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

(a) $f'(4) \approx \dfrac{f(5) - f(3)}{5 - 3} = -3$

1 : answer

(b) $\int_2^{13} (3 - 5f'(x))\, dx = \int_2^{13} 3\, dx - 5\int_2^{13} f'(x)\, dx$
$= 3(13 - 2) - 5(f(13) - f(2)) = 8$

2 : { 1 : uses Fundamental Theorem of Calculus; 1 : answer }

(c) $\int_2^{13} f(x)\, dx \approx f(2)(3 - 2) + f(3)(5 - 3) + f(5)(8 - 5) + f(8)(13 - 8) = 18$

2 : { 1 : left Riemann sum; 1 : answer }

(d) An equation for the tangent line is $y = -2 + 3(x - 5)$. Since $f''(x) < 0$ for all x in the interval $5 \le x \le 8$, the line tangent to the graph of $y = f(x)$ at $x = 5$ lies above the graph for all x in the interval $5 < x \le 8$.

Therefore, $f(7) \le -2 + 3 \cdot 2 = 4$.

An equation for the secant line is $y = -2 + \frac{5}{3}(x - 5)$.

Since $f''(x) < 0$ for all x in the interval $5 \le x \le 8$, the secant line connecting $(5, f(5))$ and $(8, f(8))$ lies below the graph of $y = f(x)$ for all x in the interval $5 < x < 8$.

Therefore, $f(7) \ge -2 + \frac{5}{3} \cdot 2 = \frac{4}{3}$.

4 : { 1 : tangent line; 1 : shows $f(7) \le 4$; 1 : secant line; 1 : shows $f(7) \ge \frac{4}{3}$ }

5 5 5 5 5 5 5 5 5 5

NO CALCULATOR ALLOWED

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

5A1

Work for problem 5(a)

$$f'(4) \approx \frac{-2-4}{5-3} = \frac{-6}{2} = -3$$

Work for problem 5(b)

$$\int_2^{13} 3 - 5f'(x)\,dx = \int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$$

$$3x\Big|_2^{13} - 5\left(f(13) - f(2)\right)$$

$$33 - 5(6-1)$$

$$33 - 5(5)$$

$$33 - 25 = 8$$

Do not write beyond this border.

5 5 5 5 5 5 5 5 5 5

NO CALCULATOR ALLOWED

5A2

Work for problem 5(c)

$(3-2)\cdot 1 + (5-3)\cdot 4 + (8-5)\cdot -2 + (13-8)\cdot 3$

$1+8-6+15=18$

Work for problem 5(d)

$y+2=3(x-5)$

$y+2=3(7-5)$

$y+2=3(2)$

$y=4$

since $f''(x)<0$ the tangent line is an overapproximation so $f(7)\leq 4$

$\frac{3--2}{8-5}=\frac{5}{3}$

$y+2=\frac{5}{3}(x-5)$

$y+2=\frac{5}{3}(7-5)$

$y+2=\frac{5}{3}(2)$

$y+2=\frac{10}{3}$

$y=\frac{10}{3}-2$

$y=\frac{4}{3}$

since $f''(x)<0$ the secant line is an underapproximation at $f(7)$, so $f(7)$ must be $\geq \frac{4}{3}$

Do not write beyond this border.

5 5 5 5 5 5 5 5 5 5

NO CALCULATOR ALLOWED

5B1

x	2	3	5	8	13
$f(x)$	1	4	−2	3	6

Do not write beyond this border.

Work for problem 5(a)

$$f'(4) = \frac{f(5)-f(3)}{5-3}$$

$$f'(4) = \frac{-2-4}{5-3}$$

$$f'(4) = \frac{-6}{2} = -3$$

Work for problem 5(b)

$$\int_2^{13} (3 - 5f'(x))\,dx$$

$$\int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$$

$$3x\Big|_2^{13} - 5\left(f(x)\Big|_2^{13}\right)$$

$$3(13) - 3(2) - 5(f(13) - f(2))$$

$$26 - 6 - 5(6-1)$$

$$20 - 5(5)$$

$$20 - 25$$

$$\int_2^{13} (3 - 5f'(x))\,dx = 5$$

NO CALCULATOR ALLOWED

5B2

Work for problem 5(c)

$\int_2^{13} f(x)\,dx$ using left Riemann sums:

$(1\cdot 2)+(2\cdot 4)+(3\cdot -2)+(5\cdot 3)$

$2+8-6+15=19$

$\int_2^{13} f(x)\,dx \approx 19$

Work for problem 5(d)

$f'(5)=3$ and $f(5)=-2$

$y+2=3(x-5)$

$y=3x-15-2$

$y=3x-17$

at $x=7$: $y=3(7)-17$

$y=21-17$

$y=4$

Because $f''(x)<0$, $f'(x)$ is decreasing over the interval $5\le x\le 8$. This means $f'(5)$ is the largest value over this interval, so $f(7)$ can not be any greater than 4. Therefore, $f(7)\le 4$.

$f'(c)=\frac{f(8)-f(5)}{8-5}$

$f'(c)=\frac{3--2}{3}=\frac{5}{3}$

$y+2=\frac{5}{3}(x-5)$

at $x=7$: $y+2=\frac{5}{3}(7-5)$

$y+2=\frac{10}{3}$

$y=\frac{4}{3}$

Because this secant line is the average slope of the interval $5\le x\le 8$, it is an underapproximation of $f(7)$. Therefore $f(7)\ge\frac{4}{3}$.

Do not write beyond this border.

NO CALCULATOR ALLOWED

5C$_1$

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

Work for problem 5(a)

$f'(4) = \frac{-2-1}{5-2} = -\frac{3}{3} = -1$

$\boxed{f'(4) = -1}$

Work for problem 5(b)

$\int_2^{13} (3 - 5f'(x))\,dx$

$\Big|_2^{13} 3x - 5(f(x))$

$13(3) - 6$

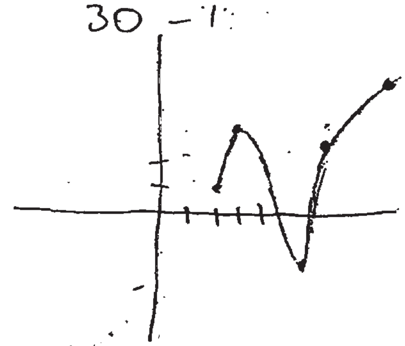
$39 - 6 = 33$

$33 - 29 = 4$

$\Big|_2^{13} 5f(x)$

$5(f(13)) - 5(f(2))$

$30 - 1$

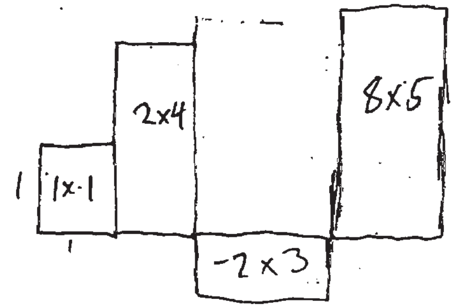


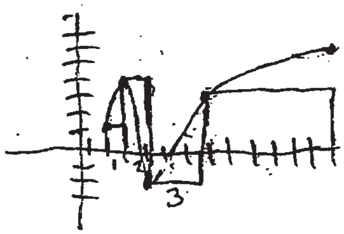
Do not write beyond this border.

NO CALCULATOR ALLOWED

5C8

Work for problem 5(c)



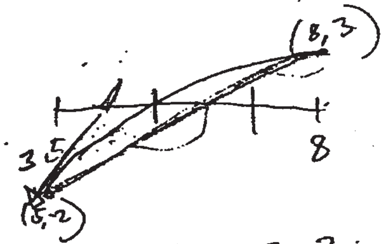


1+8 −6 + 40

9 − 6 43

Work for problem 5(d)

$f'(5) = 3$



$y + 2 = 3(x - 5)$

$y + 2 = 3x - 15$

$y = 3x - 17$

$y = 3(7) - 17$

$y = 21 - 17$

$y = 4 \leq 4$

$f(7)$

$\frac{3+2}{8-5} = \frac{5}{3}$

$y + 2 = \frac{5}{3}(x - 5)$

$y + 2 = \frac{5x}{3} - \frac{25}{3}$

$f(7) \geq \frac{4}{3}$

$y = \frac{5x}{3} - \frac{31}{3}$

$y = \frac{35}{3} - \frac{31}{3} = \frac{4}{3} \geq \frac{4}{3}$

Do not write beyond this border.

Question 5

Overview

This problem presented students with a table of values for a function f sampled at five values of x. It was also stated that f is twice differentiable for all real numbers. Part (a) asked for an estimate for $f'(4)$. Since $x = 4$ falls between the values sampled on the table, students should have calculated the slope of the secant line to the graph of f corresponding to the closest pair of points in the supplied data that brackets $x = 4$. Part (b) tested students' ability to apply properties of the definite integral to evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Part (c) asked for an approximation to $\int_2^{13} f(x)\,dx$ using the subintervals of $[2, 13]$ indicated by the data in the table. In part (d) it was also stated that $f'(5) = 3$ and $f''(x) < 0$ for all x in $[5, 8]$. Students were asked to use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$ and to use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$. For the former inequality, students should have used the fact that f'' is negative (so f' is decreasing) on $[5, 8]$ so that the tangent line at $x = 5$ lies above the graph of f throughout $(5, 8]$. For the latter inequality, students should have used the sign of f'' to conclude that the indicated secant line lies below the graph of f for $5 < x < 8$; in particular, the point on the graph of the secant line corresponding to $x = 7$ is below the corresponding point on the graph of f.

Sample: 5A
Score: 9

The student earned all 9 points. In part (b) the student's second line earned the first point, and the third line earned the second point. In part (c) the student's first line earned both points. In part (d) the student's first line earned the first point. The second point was earned by showing that $y = 4$ when $x = 7$ on the tangent line, stating the desired inequality $f(7) \le 4$, and giving an acceptable reason to validate the inequality. The third and fourth points were earned in a similar manner using the secant line.

Sample: 5B
Score: 6

The student earned 6 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the student's second line earned the point. In part (b) the student's fourth line earned the first point for use of the Fundamental Theorem of Calculus. The student makes subsequent errors. In part (c) the student's second line earned the first point since seven of the eight presented factors are correct. The student did not earn the answer point. In part (d) the student's second line on the left earned the first point. The second point was earned by showing that $y = 4$ when $x = 7$ on the tangent line, stating the desired inequality $f(7) \le 4$, and giving an acceptable reason to validate the inequality. The student's third line on the right earned the third point. The last point was not earned since the student's reason does not validate the inequality $f(7) \ge \frac{4}{3}$.

Question 5 (continued)

Sample: 5C
Score: 4

The student earned 4 points: no points in part (a), 1 point in part (b), 1 point in part (c), and 2 points in part (d). In part (a) the student's answer is incorrect. In part (b) the student earned the first point by correctly applying the Fundamental Theorem of Calculus to the derivative of *f*. The student makes a subsequent arithmetic error. In part (c) the student earned the first point since seven of the eight presented factors are correct. The student did not earn the answer point. In part (d) the student earned the first and third points for correct equations for the tangent and secant lines. Since the student does not explain why either of the two inequalities is valid, the student did not earn the other points.

AP® CALCULUS BC
2009 SCORING GUIDELINES

Question 6

The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

(b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

(a) $1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots$	$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$
(b) $1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24} + \cdots + \frac{(x-1)^{2n}}{(n+1)!} + \cdots$	$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$
(c) $\lim\limits_{n\to\infty} \left\lvert \dfrac{\frac{(x-1)^{2n+2}}{(n+2)!}}{\frac{(x-1)^{2n}}{(n+1)!}} \right\rvert = \lim\limits_{n\to\infty} \frac{(n+1)!}{(n+2)!}(x-1)^2 = \lim\limits_{n\to\infty} \frac{(x-1)^2}{n+2} = 0$ Therefore, the interval of convergence is $(-\infty, \infty)$.	$3: \begin{cases} 1: \text{sets up ratio} \\ 1: \text{computes limit of ratio} \\ 1: \text{answer} \end{cases}$
(d) $f''(x) = 1 + \frac{4 \cdot 3}{6}(x-1)^2 + \frac{6 \cdot 5}{24}(x-1)^4 + \cdots + \frac{2n(2n-1)}{(n+1)!}(x-1)^{2n-2} + \cdots$ Since every term of this series is nonnegative, $f''(x) \geq 0$ for all x. Therefore, the graph of f has no points of inflection.	$2: \begin{cases} 1: f''(x) \\ 1: \text{answer} \end{cases}$

NO CALCULATOR ALLOWED

Work for problem 6(a)

$$1+(x-1)^2+\frac{(x-1)^4}{2}+\frac{(x-1)^6}{6}+\cdots+\frac{(x-1)^{2n}}{n!}+\cdots$$

Do not write beyond this border.

Work for problem 6(b)

$(x-1)^2$

$$1+\frac{(x-1)^2}{2}+\frac{(x-1)^4}{6}+\frac{(x-1)^6}{24}+\cdots+\frac{(x-1)^{2n}}{(n+1)!}+\cdots$$

Do not write beyond this border.

Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

Work for problem 6(c)

$$\lim_{n\to\infty}\left|\frac{(x-1)^{2(n+1)}}{(n+2)!}\cdot\frac{(n+1)!}{(x-1)^{2n}}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{2n+2}}{(n+2)(x-1)^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-1)^2}{(n+2)}\right| = \lim_{n\to\infty}\left|\frac{x^2-2x+1}{n+2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{x^2}{n}-\frac{2x}{n}+\frac{1}{n}}{1+\frac{2}{n}}\right| = \lim_{n\to\infty}\frac{0}{1} = 0<1 \quad \boxed{(-\infty,\infty)}$$

converges for all x.

Work for problem 6(d)

$$f'(x) = (x-1)+\frac{2}{3}(x-1)^3+\frac{1}{4}(x-1)^5+\cdots+\frac{2(n-1)}{(n+1)!}(x-1)^{2n-1}+\cdots$$

$$f''(x) = 1+2(x-1)^2+\frac{5}{4}(x-1)^4+\cdots+\frac{2n(2n-1)}{(n+1)!}(x-1)^{2n-2}+\cdots$$

$f''(x)$ is always positive so f has no points of inflection.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

Work for problem 6(a)

$$P_4(x) = 1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \frac{(x-1)^8}{24}$$

Work for problem 6(b)

$$f(x) = \frac{1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-\cancel{6})^6}{6} + \frac{(x-1)^8}{24} - 1}{(x-1)^2}$$

$$= \frac{(x-1)^2}{(x-1)^2} + \frac{(x-1)^4}{2(x-1)^2} + \frac{(x-1)^6}{6(x-1)^2} + \frac{(x-1)^8}{24(x-1)^2}$$

$$= 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24}$$

general term: $\sum_{n=1}^{\infty} \frac{(x-1)^{2n-2}}{n!}$

Do not write beyond this border. Do not write beyond this border.

Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

Work for problem 6(c)

4

$$\lim_{n\to\infty}\left|\frac{(x-1)^{2n}}{(n+1)!}\cdot\frac{n!}{(x-1)^{2n-2}}\right| = (x-1)^2\lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right|$$

$$=\left|(x-1)^2\right|\lim_{n\to\infty}\left|\frac{n!}{(n+1)n!}\right|$$

$$=\left|(x-1)^2\right|\lim_{n\to\infty}\frac{1}{n+1}$$

$$=\left|(x-1)^2\right|\cdot 0$$

interval of convergence $(-\infty,\infty)$

Do not write beyond this border.

Do not write beyond this border.

Work for problem 6(d)

$$P_4(x)=1+\frac{(x-1)^2}{2}+\frac{(x-1)^4}{6}+\frac{(x-1)^6}{24}$$

$$P_4'(x)=\frac{2(x-1)}{2}+\frac{4(x-1)^3}{6}+\frac{6(x-1)^5}{24}+8$$

$$=(x-1)+\frac{2(x-1)^3}{3}+\frac{(x-1)^5}{4}$$

$$P_4''(x)=\frac{2}{3}+\frac{6(x-1)^2}{3}+\frac{5(x-1)^4}{4}$$

$$0=\frac{2}{3}+2(x-1)^2+\frac{5(x-1)^4}{4}$$

$$-\frac{2}{3}=(x-1)^2(2+\frac{5}{4}(x-1)^2)$$

GO ON TO THE NEXT PAGE.

6 6 6 6 6 6 6 6 6 6 6C₁

NO CALCULATOR ALLOWED

Work for problem 6(a)

$e^x = \sum \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$e^{(x-1)^2} = 1 + (x-1)^2 + \frac{((x-1)^2)^2}{2!} + \frac{((x-1)^2)^3}{3!} + \ldots$

$$\boxed{e^{(x-1)^2} = 1 + (x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \ldots \frac{(x-1)^{2n}}{n!} + \ldots}$$

Do not write beyond this border.

Do not write beyond this border.

Work for problem 6(b)

$$\frac{\left(\cancel{1} + (x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \ldots\right) \cancel{-1}}{(x-1)^2} = \frac{\cancel{(x-1)^2} + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \ldots}{\cancel{(x-1)^2}}$$

$$= \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \frac{(x-1)^8}{4!} + \ldots$$

$$T(x) = \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \frac{(x-1)^8}{4!} + \ldots \frac{(x-1)^{2n+2}}{(n+2)!} + \ldots$$

NO CALCULATOR ALLOWED

Work for problem 6(c)

$2n+2+2 \quad 2n+4$

$$\left| \frac{(x-1)^{2(n+1)+2}}{\cancel{(n+3)}!} \cdot \frac{\cancel{(n+2)}!}{(x-1)^{2n+2}} \right|$$

$$= (x-1)^2 \cdot \lim_{n\to\infty} \frac{1}{n+3}$$

$$= (x-1)^2 \cdot \infty$$

interval = $\boxed{-\infty < x < \infty}$

Work for problem 6(d)

$f''(1) > 0$ and $f'(1)$ is zero based on taylor series

$\therefore$ since $f''(1) > 0$ and $f'(1) = 0$, there is a point of inflection at $x=1$ by second derivative test.

Do not write beyond this border.

GO ON TO THE NEXT PAGE

AP® CALCULUS BC
2009 SCORING COMMENTARY

Question 6

Overview

This problem reminded students of the Maclaurin series for e^x and defined a function f by $f(x) = \dfrac{e^{(x-1)^2} - 1}{(x-1)^2}$ if $x \neq 1$ and $f(1) = 1.$ It was noted that f is continuous and has derivatives of all orders at $x = 1.$ Part (a) asked for the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$. Students could have found this by substituting $(x-1)^2$ for x in the Maclaurin series for e^x. In part (b) students needed to manipulate the Taylor series from part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1.$ Part (c) asked students to apply the ratio test to determine the interval of convergence for the Taylor series found in part (b). Part (d) asked students to use the Taylor series from part (c) to determine whether f has any points of inflection. Students needed to differentiate the Taylor series term-by-term twice; then they should have concluded that the resulting series for f'' is nonnegative for all x, from which it follows that the graph of f has no points of inflection.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 1 point in part (a), 2 points in part (b), 3 points in part (c), and no points in part (d). In part (a) the student earned the first point since the first five terms are correct. The student does not include a general term. In parts (b) and (c) the student's work is correct. In part (d) the student's second derivative is incorrect.

Sample: 6C
Score: 4

The student earned 4 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the response is missing the first term of the first four nonzero terms of the series, so the first point was not earned. The student's general term is correct and earned the point. In part (c) the student earned the first point for the correct ratio. The limit is computed incorrectly. Although the interval of convergence provided is the correct answer, it does not follow from the student's limit. In part (d) the student does not find a series for $f''(x)$.